

# **PolyBounds**

## User's Manual

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# Chapter 1

## Overview

The program calculates bounds for real and complex polynomial zeros according to the results found in the following papers (total 25 estimates):

- Dehmer M. (2006) On the location of zeros of complex polynomials. *Journal of Inequalities in Pure and Applied Mathematics*, Vol 7 (1)
- Dehmer M., Mowshowitz A. (2011) Bounds on the moduli of polynomial zeros. *Applied Mathematics and Computation*, accepted, 218: 4128-4137.
- Joyal A., Labelle G., Rahman Q.I. (1967) On the location of polynomials. *Canadian Mathematical Bulletin* 10: 53-63.
- Jain V.K. (1986) On the zeros of polynomials II. *Journal of Mathematical and Physical Sciences* 20: 259-267.
- Kalantari B. (2005) An infinite family of bounds on zeros of analytic functions and relationship to Smale's bound. *Mathematics of Computation* 74: 841-852.
- Kojima J. (1914) On a theorem of Hadamard and its applications. *Tohoku Mathematical Journal* 5: 54-60.
- Marden M. (1966) *Geometry of polynomials*. Mathematical Surveys of the American Mathematical Society, Vol. 3. Rhode Island, USA.

### 1.1 Polynomials

The polynomials, for which bounds are computed, are generated at random using the following distributions:

- Gaussian.
- Poisson.
- Geometric.

- Uniform.
- Uniform in  $[-1, +1]$ .

The following special class polynomials can be generated:

- Real Lacunary-1:  $z^n - z^{n-1} - a_1 * z + a_0, a_1, a_0 > 0$ , with Gaussian coefficients.
- Real Lacunary-2:  $z^n - a_1 * z + a_0, a_1, a_0 > 0$ , with Gaussian coefficients.
- Complex Lacunary-1:  $z^n - z^{n-1} - a_1 * z + a_0, a_1 * a_0! = 0$ , with Gaussian coefficients.
- Complex Lacunary-2:  $z^n - a_1 * z + a_0, a_1 * a_0! = 0$ , with Gaussian coefficients.
- Complex Constrained-1:  $|a_i| < 1$ , with Gaussian coefficients.
- Complex Constrained-2:  $|a_i| < 1, a_n$  - arbitrary, with Gaussian coefficients.
- Complex Constrained-3:  $|a_i| < 1, a_n, a_{n-1}$  - arbitrary, with Gaussian coefficients.
- Complex Constrained-4:  $|a_i|/|a_n| < 1$ , with Gaussian coefficients.
- Complex Constrained Multiple:  $f = f_1 * f_2, |c_{nj}| > |c_i|$ , where  $f_1$  and  $f_2$  are complex polynomials with Gaussian coefficients.
- Complex Multiple:  $f = f_1 * f_2$ , where  $f_1$  and  $f_2$  are complex polynomials with Gaussian coefficients.

## 1.2 System Requirements

Hardware: Intel Pentium III or AMD Athlon or higher, RAM 512, 5MB HDD, keyboard, mouse.

Software: OS Microsoft Windows XP/Vista/7. Microsoft .NET Framework 3.5 or higher.

## 1.3 Acknowledgements

The author is thankful to Prof. Dr. Matthias Dehmer<sup>1</sup> for his helpful comments on the program.

## 1.4 Contacts

For bugs and issues, please contact Yury Tsoy: yurytsoy@gmail.com

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<sup>1</sup><http://www.dehmer.org>

## Chapter 2

# Usage of the Program

### 2.1 Main Window

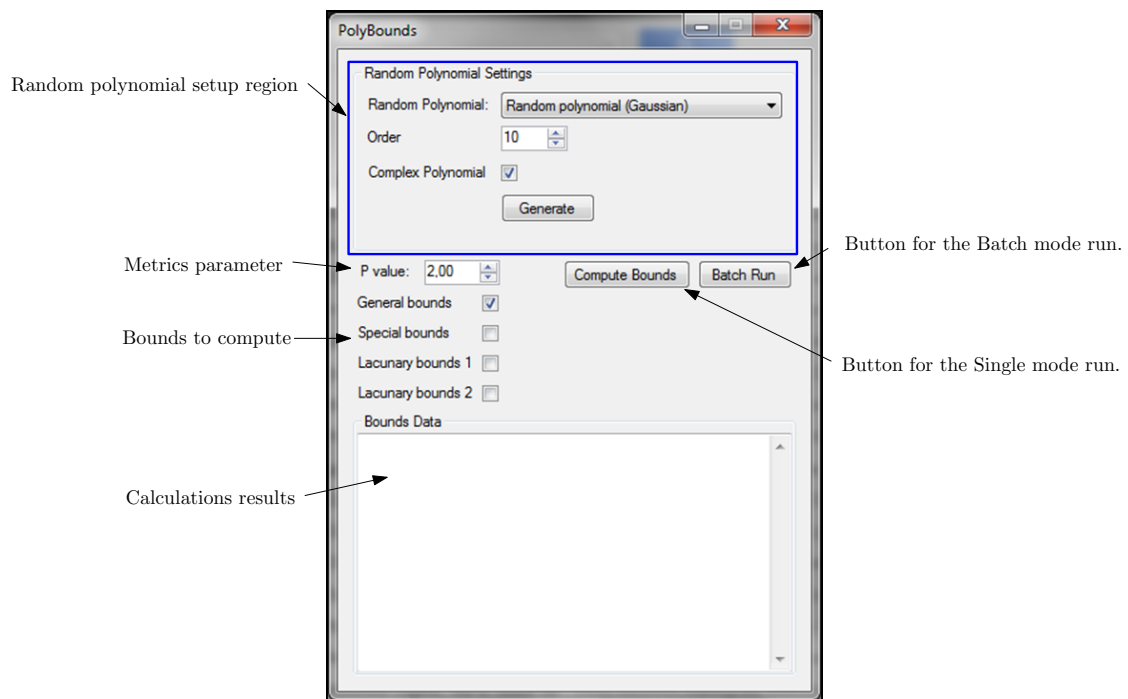


Figure 2.1: Main Window

### 2.2 Computing the polynomial bounds

There are two modes for computing the zero bounds:

1. The first mode concerns processing of a single random polynomial with possibility for selection of the distribution for polynomial coefficients.
2. Batch mode, which involves calculation of all available bounds for specified number of random polynomials.

See more detailed descriptions of each mode below.

## 2.3 Setting up random polynomial coefficients

Before description of the two modes mentioned above let's consider a step, which is obligatory for both modes. This step considers selection of parameters for random polynomial (or set of polynomials for the Batch mode). Region for setting polynomial parameters is shown by fig. 2.2.

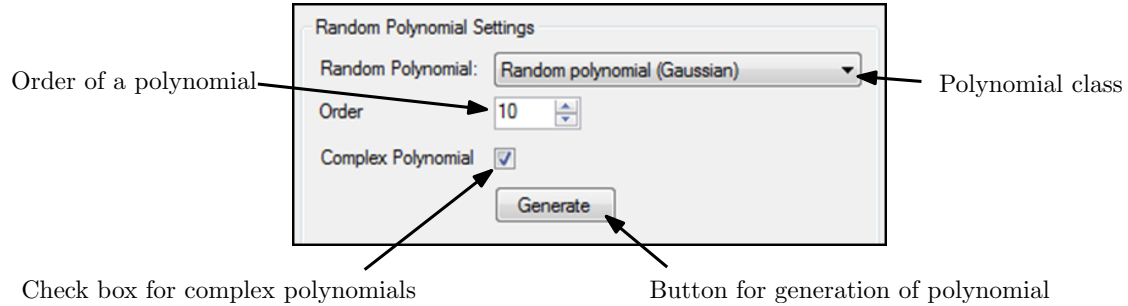


Figure 2.2: Polynomial setup.

For a random polynomial order and distribution for polynomial coefficients can be set up. The following distributions are supported:

- Gaussian.
- Poisson.
- Geometric.
- Uniform.
- Uniform on interval  $[-1, 1]$ .

For all distributions except of the last one, there is a possibility to set parameters, like mean, std. deviation etc. See sections describing single and batch modes.

**Note:** Since different bounds require complex or real polynomials, then both types of polynomials would be generated, but if "Complex Polynomial" is checked then a complex polynomial will have non-zero imaginary part and real polynomial will have only real parts of coefficients, while otherwise complex and real polynomials would in fact be the same.

### 2.3.1 Computing zero bounds in the Single Mode

In this mode the user should select random polynomial parameters and a special parameter  $p$ , which is required for several bounds, and then all available bounds are calculated for this random polynomial.

**Note:** Calculation results substitute current contents of the text box, so be sure that you've saved all valuable data before starting calculations.

The calculation of polynomials zero bounds the Single Mode consists of the following steps:

1. Set up random polynomial parameters (order, distribution and type of coefficients).
2. Select  $p$  parameter, required for some bounds.
3. Press the "Compute Bounds" button.
4. Set up parameters for the selection random distribution.

The result is shown in text box in a lower part of the main window and is automatically saved in a log-file, which is placed in the program directory. The name of log-file consists of current date, time and order of polynomial. For example file "10.07.2011\_22.41.09\_5.log" was created on July, 10th at 22 : 41 : 09 for 5th order polynomial.

Log-file contents include the following sections:

- polynomial order.
- selected distribution.
- generated complex and real polynomial coefficients in the order  $c_0, c_1, \dots, c_n$ .
- value of the  $p$  parameter.
- sharpest and widest bounds for generated polynomial with their values.
- list of all bounds' values sorted by rank in ascending order (from best to worst).

**Note:** Some bounds require obligatory conditions to be satisfied (like existence of two positive roots for a special polynomial). If these conditions check is failed then the bounds are considered infinite, which affects the results.

### 2.3.2 Computing zero bounds in the Batch Mode

In this mode the user should select random polynomial parameters and a special parameter  $p$ , which is required for several bounds, and then all available bounds are calculated for this random polynomial for a specified number of runs. At each run a new random polynomial is generated.



Overall results are calculated via computation of sum rank for all bounds so that the bounds with the lowest sum rank happened to be the sharpest, while the bounds with the largest sum rank is the widest.

**Note:** Calculation results substitute current contents of the text box, so be sure that you've saved all valuable data before starting calculations.

The calculation of polynomials zero bounds for the Batch Mode consists of the following steps:

1. Set up random polynomial parameters (order, distribution and type of coefficients).
2. Select  $p$  parameter, required for some bounds.
3. Press the "Batch Run" button.
4. Specify number of runs.
5. Set up parameters for the selection random distribution.

The overall results are shown in text box in a lower part of the main window and are automatically saved in a log-file, which is saved in a subdirectory of the program directory along with log-files for each single polynomial zero bounds, which were generated during a batch run. The name of subdirectory consists of current date, time and order of polynomial. For example subdirectory "10\_10.07.2011\_21.39.15" was created on July, 10th at 21 : 39 : 15 for 10th order polynomial.

Log-file for overall results is always named "\_total.log" and includes the following sections:

- polynomial order.
- selected distribution.
- number of independent runs.
- value of  $p$  parameter.
- sharpest and widest bounds for generated polynomial with their sum ranks.
- list of all bounds sorted in ascending order by sum ranks.

**Note:** Some bounds require obligatory conditions to be satisfied (like existence of two positive roots for a special polynomial). If these conditions check is failed then the bounds are considered infinite, which affects the results.

## Chapter 3

### Zero bounds

The correspondence between zero bounds as they are referred in the log-files and the theorems is the following<sup>1</sup>.

#### 3.1 Cauchy, Th. (1)

**Theorem 1 (Cauchy [8])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \dots, n$$

*be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk  $|z| \leq 1 + M$ , where*

$$M := \max_{0 \leq j \leq n-1} \frac{|a_j|}{|a_n|}.$$

#### 3.2 Joyal, Th. (2)

**Theorem 2 (Joyal [4])** *For  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , all zeros of*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad a_i \in \mathcal{C}, \quad i = 0, 1, \dots, n$$

*lie in*

$$|z| \leq \left[ \frac{1}{2} \left\{ 1 + \sqrt{1 + 4M_p^q} \right\} \right]^{\frac{1}{q}}, \quad (3.1)$$

*where*

$$M_p = \left( \sum_{k=1}^n \left| \frac{a_{n-1} a_{n-k} - a_n a_{n-k-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}}, \quad a_{-1} = 0. \quad (3.2)$$

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<sup>1</sup> $K(0, \delta)$  denotes a dish in the complex plane with center at 0 and radius  $\delta$ .

### 3.3 Mohammad, Th. (3)

**Theorem 3 (Mohammad [3])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \dots, n$$

*be a complex polynomial. All zeros  $f(z)$  lie in*

$$|z| \leq 2 \max \left( \frac{|a_k|}{|a_{k+1}|} \right), \quad 0 \leq k \leq n-1. \quad (3.3)$$

### 3.4 Kojima, Th. (4)

**Theorem 4 (Kojima [6])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \dots, n$$

*be a complex polynomial. All zeros  $f(z)$  lie in*

$$|z| \leq \max \left( \frac{|a_0|}{|a_1|}, 2 \frac{|a_k|}{|a_{k+1}|} \right), \quad 1 \leq k \leq n-1. \quad (3.4)$$

### 3.5 Jain, Th. (5)

**Theorem 5 (Jain [3])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \dots, n$$

*be a complex polynomial. All zeros  $f(z)$  lie in*

$$|z| \leq \frac{\max \left( \frac{|a_{n-1}|}{|a_n|}, 2 \frac{|a_{n-2}|}{|a_{n-1}|}, 3 \frac{|a_{n-3}|}{|a_{n-2}|}, \dots, n \frac{|a_0|}{|a_1|} \right)}{\ln(2)}. \quad (3.5)$$

### 3.6 Kuniyeda, Th. (6)

**Theorem 6 (Kuniyeda [7])** *Let  $p, q > 1$  mit  $\frac{1}{p} + \frac{1}{q} = 1$ . All zeros of*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad a_i \in \mathcal{C}, \quad i = 0, 1, \dots, n$$

*lie in*

$$|z| \leq \left\{ 1 + \left[ \sum_{i=0}^{n-1} \left| \frac{a_i}{a_n} \right|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \quad (3.6)$$

### 3.7 Kuniyeda, Th. (7)

**Theorem 7 (Kuniyeda [7])** For  $p > 0$ , all zeros of

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad a_i \in \mathcal{C}, \quad i = 0, 1, \dots, n$$

lie in

$$|z| \leq \left\{ \frac{1}{|a_n|^{\frac{p+1}{p}}} \left( \sum_{i=1}^n |a_{n-i}|^{1+p} \right)^{\frac{1}{p}} + 1 \right\}^{\frac{p}{p+1}}. \quad (3.7)$$

### 3.8 Joyal, Th. (8)

**Theorem 8 (Joyal [4])** For  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , all zeros of

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad a_i \in \mathcal{C}, \quad i = 0, 1, \dots, n$$

lie in

$$|z| \leq \left[ \frac{1}{2} \left\{ 1 + \sqrt{1 + 4M_p^q} \right\} \right]^{\frac{1}{q}}, \quad (3.8)$$

where

$$M_p = \left( \sum_{k=1}^n \left| \frac{a_{n-1}a_{n-k} - a_n a_{n-k-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}}, \quad a_{-1} = 0. \quad (3.9)$$

### 3.9 Dehmer, Th. (9)

**Theorem 9 (Dehmer [2])** Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk

$$K \left( 0, \frac{1 + \phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + 4M_1}}{2} \right), \quad (3.10)$$

where

$$\phi_2 := \left| \frac{a_{n-1}}{a_n} \right|. \quad (3.11)$$

### 3.10 Dehmer, Th. (10)

**Theorem 10 (Dehmer [2])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

*be a complex polynomial. Suppose that  $\phi_2 := \left| \frac{a_{n-1}}{a_n} \right|$  and*

$$|a_j| < 1, \quad 0 \leq j \leq n-2. \quad (3.12)$$

*All zeros of  $f(z)$  lie in the closed disk*

$$K \left( 0, \frac{1 + \phi_2}{2} + \frac{\sqrt{(\phi_2 - 1)^2 + \frac{4}{|a_n|}}}{2} \right). \quad (3.13)$$

*The bound is sharp for all polynomials of the form*

$$f(z) = az^n - bz^{n-1} - [z^{n-2} + \cdots + z + 1], \quad a, b > 0. \quad (3.14)$$

### 3.11 Cauchy, Th. (11)

**Theorem 11 (Cauchy [8])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \dots, n$$

*be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk  $K(0, \rho_C)$ , where  $\rho_C$  denotes the positive zero of*

$$H_C(z) := |a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n. \quad (3.15)$$

### 3.12 Dehmer, Th. (12)

**Theorem 12 (Dehmer [1])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0, \quad (3.16)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk  $K(0, \max(1, \delta))$  where  $\delta$  denotes the positive root of the equation*

$$z^{n+1} - (1 + M_2)z^n + M_2 = 0, \quad (3.17)$$

*and*

$$M_2 := \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|. \quad (3.18)$$

*The bound is sharp for all polynomials of the form*

$$f(z) = az^n - b[z^{n-1} + \cdots + z + 1], \quad a, b > 0. \quad (3.19)$$

### 3.13 Dehmer, Th. (13)

**Theorem 13 (Dehmer [2])** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

*be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk  $K(0, \max(1, \delta))$  where  $\delta$  denotes the positive root of the equation*

$$z^{n+1} - \left(1 + \left|\frac{a_{n-1}}{a_n}\right|\right) z^n + \left(\left|\frac{a_{n-1}}{a_n}\right| - M_1\right) z^{n-1} + M_1 = 0, \quad (3.20)$$

*and*

$$M_1 := \max_{0 \leq j \leq n-2} \left|\frac{a_j}{a_n}\right|, \quad (3.21)$$

*The bound is sharp for all polynomials of the form*

$$f(z) = az^n - bz^{n-1} - c[z^{n-2} + \cdots + z + 1], \quad a, b > 0, c \geq 0. \quad (3.22)$$

### 3.14 Dehmer, Th. (14)

**Theorem 14 (Dehmer [2])** *Let*

$$M_3 := \max_{2 \leq j \leq n} \left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|, \quad a_{-1} := 0, \quad (3.23)$$

*and*

$$\phi_1 := \frac{|a_{n-1}^2 - a_n a_{n-2}|}{|a_n|^2}, \quad (3.24)$$

*In addition, let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

*be a complex polynomial. All zeros of  $f(z)$  lie in the closed disk  $K(0, \delta)$  where  $\delta > 1$  is the largest positive root of the equation*

$$z^3 - z^2 - (M_3 + \phi_1)z + \phi_1 = 0. \quad (3.25)$$

*Moreover,*

$$1 < \delta < 1 + \sqrt{M_3 + \phi_1}. \quad (3.26)$$

### 3.15 Dehmer, Th. (15)

**Theorem 15 (Dehmer [2])** *Let*

$$f(z) = z^n - a_1 z + a_0, \quad a_1 a_0 \neq 0, \quad n > 2, \quad (3.27)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in  $K(0, \max(1, \delta))$ , where  $\delta$  is the unique positive root of the equation*

$$z^n - |a_1|z - |a_0| = 0. \quad (3.28)$$

### 3.16 Dehmer, Th. (16)

**Theorem 16 (Dehmer [2])** *Let*

$$f(z) = z^n - a_1 z + a_0, \quad a_1 a_0 \neq 0, \quad n > 2, \quad (3.29)$$

*be a polynomial with arbitrary coefficients. All zeros of  $f(z)$  lie in  $K(0, \max(1, \delta))$ , where  $\delta$  is the unique positive root of the equation*

$$z^n - M_4 z - M_4 = 0. \quad (3.30)$$

### 3.17 Dehmer Bounds Lacunary (1)

**Theorem 17 (Dehmer [2])** *If the real polynomial*

$$f(z) = z^n - z^{(n-1)} - a_1 z + a_0, \quad a_1 a_0 > 0, \quad n > 2, \quad (3.31)$$

*has two positive zeros, its largest positive zero  $\delta$  satisfies*

$$\delta < 1 + \sqrt{a_1}. \quad (3.32)$$

### 3.18 Dehmer Bounds Lacunary (3)

**Theorem 18 (Dehmer [2])** *If the real polynomial*

$$f(z) = z^n - a_1 z + a_0, \quad a_1, a_0 > 0, \quad n > 2, \quad (3.33)$$

*has two positive zeros, its largest positive zero  $\delta$  satisfies*

$$\delta < \frac{1}{2} + \frac{\sqrt{4a_1 + 1}}{2}. \quad (3.34)$$

### 3.19 Dehmer Bounds Lacunary (5)

**Theorem 19 (Dehmer [2])** *Let*

$$f(z) = z^n - z^{n-1} - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.35)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in  $K(0, \delta)$ , where  $\delta > 1$  is the largest positive root of the equation*

$$z^{n+1} - 2z^n - |a_1|z^2 + (|a_1| - |a_0|)z + |a_0| = 0. \quad (3.36)$$

### 3.20 Dehmer Bounds Lacunary (6)

**Theorem 20 (Dehmer [2])** *Let*

$$M_4 := \max(|a_1|, |a_0|), \quad (3.37)$$

*and let*

$$f(z) = z^n - z^{n-1} - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.38)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in  $K(0, \delta)$ , where  $\delta > 1$  is the largest positive root of the equation*

$$z^{n+1} - 2z^n - M_4 z^2 + M_4 = 0. \quad (3.39)$$

### 3.21 Dehmer Bounds Lacunary (7)

**Theorem 21 (Dehmer [2])** *Let*

$$f(z) = z^n - z^{n-1} - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.40)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in*

$$K(0, 1 + |a_1| + |a_0|). \quad (3.41)$$

### 3.22 Dehmer Bounds Lacunary (8)

**Theorem 22 (Dehmer [2])** *Let*

$$f(z) = z^n - z^{n-1} - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.42)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in*

$$K(0, \frac{1}{2} + \frac{\sqrt{1 + 4|a_1| + 4|a_0|}}{2}). \quad (3.43)$$

### 3.23 Dehmer Bounds Lacunary (9)

**Theorem 23 (Dehmer [2])** *Let*

$$f(z) = z^n - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.44)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in  $K(0, \max(1, \delta))$ , where  $\delta$  is the unique positive root of the equation*

$$z^n - |a_1|z - |a_0| = 0. \quad (3.45)$$



### 3.24 Dehmer Bounds Lacunary (10)

**Theorem 24 (Dehmer [2])** *Let*

$$f(z) = z^n - a_1 z + a_0, \quad a_1, a_0 \neq 0, \quad n > 2, \quad (3.46)$$

*be a polynomial with arbitrary coefficients. All zeros of  $f(z)$  lie in  $K(0, \max(1, \delta))$ , where  $\delta$  is the unique positive root of the equation*

$$z^n - M_4 z - M_4 = 0. \quad (3.47)$$

### 3.25 Dehmer Bounds Lacunary (11)

**Theorem 25 (Dehmer [2])** *Let*

$$f(z) = z^n - a_1 z + a_0, \quad a_1 a_0 \neq 0, \quad n > 2, \quad (3.48)$$

*be a complex polynomial. All zeros of  $f(z)$  lie in*

$$K\left(0, \frac{|a_1|}{2} + \frac{\sqrt{|a_1|^2 + 4|a_0| + 4}}{2}\right). \quad (3.49)$$

### 3.26 Kalantari, Cor. (4.4)

**Theorem 26 (Kalantari [5])** *Let  $m \geq 2$  and let  $r_m \in [\frac{1}{2}, 1)$  be the positive root of the polynomial*

$$q(t) := t^{m-1} + t - 1. \quad (3.50)$$

*For  $m = 2$  and  $r_2 = \frac{1}{2}$ , all zeros of the complex polynomial*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

*lie in the closed disk*

$$K\left(0, 2 \cdot \max_{1 \leq k \leq n} \left(\left|\frac{a_{n-k}}{a_n}\right|\right)^{\frac{1}{k}}\right). \quad (3.51)$$

### 3.27 Kalantari, Cor. (4.5)

**Theorem 27 (Kalantari [5])** *Let  $m \geq 2$  and let  $r_m \in [\frac{1}{2}, 1)$  be the positive root of the polynomial*

$$q(t) := t^{m-1} + t - 1. \quad (3.52)$$

*For  $m = 3$  and  $r_3 = \frac{2}{\sqrt{5}+1}$ , all zeros of the complex polynomial*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_{n-1} \neq 0,$$

lie in the closed disk

$$K \left( 0, \frac{\sqrt{5} + 1}{2} \cdot \max_{2 \leq k \leq n+1} \left( \left| \frac{a_{n-1}a_{n-k+1} - a_na_{n-k}}{a_n^2} \right| \right)^{\frac{1}{k}} \right), \quad (3.53)$$

$a_{-1} := 0$ .

### 3.28 Grouping of bounds

The program groups the bounds into 4 classes, which can be seen on the Main Form:

Group name	Bounds included
General bounds	Cauchy, Th. (1), Joyal, Th. (2), Mohammad, Th. (3), Koijma, Th. (4), Jain, Th. (5), Kuniyeda, Th. (6), Kuniyeda, Th. (7), Joyal, Th. (8), Dehmer, Th. (9), Cauchy, Th. (11), Dehmer, Th. (12), Dehmer, Th. (13), Dehmer, Th. (14), Kalantari, Cor. (4.4), Kalantari, Cor. (4.5)
Special bounds	Dehmer, Th. (10)
Lacunary bounds 1	Dehmer Bounds Lacunary (1), Dehmer Bounds Lacunary (5), Dehmer Bounds Lacunary (6), Dehmer Bounds Lacunary (7), Dehmer Bounds Lacunary (8)
Lacunary bounds 2	Dehmer Bounds Lacunary (3), Dehmer, Th. (15), Dehmer, Th. (16), Dehmer, Th. (17)

Table 3.1: Bounds groups description

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